Tractability of Multi-Parametric Euler and Wiener Integrated Processes

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Abstract

We study average case approximation of Euler and Wiener integrated processes of d variables which are almost surely r_k-times continuously differentiable with respect to the k-th variable and $0 \le r_k \le r_{k+1}$. Let $n(\varepsilon, d)$ denote the minimal number of continuous linear functionals which is needed to find an algorithm that uses n such functionals and whose average case error improves the average case error of the zero algorithm by a factor ε. Strong polynomial tractability means that there are nonnegative numbers C and p such that

$$n(\varepsilon, d) \le C\varepsilon^{-p} \quad \text{for all} \quad d \in \mathbb{N} = \{1, 2, \dots\}, \quad \text{and} \quad \varepsilon \in (0, 1).$$

We prove that the Wiener process is much more difficult to approximate than the Euler process. Namely, strong polynomial tractability holds for the Euler case iff

$$\liminf_{k\to\infty} \frac{r_k}{\ln\, k} > \frac{1}{2 \ln\, 3},$$

whereas it holds for the Wiener case iff

$$\liminf_{k\to\infty} \frac{r_k}{k^s} > 0 \quad \text{for some} \quad s > \tfrac{1}{2}.$$

Other types of tractability are also studied.

1 Introduction

Tractability of multivariate problems has been recently an active research area. The reader may see the current progress on tractability in [10, 11, 12]. Tractability has been studied in various settings and for various error criteria.

This field deals with problems defined on spaces of d-variate functions. For many practical computational problems d is large. This holds for problems in mathematical finance, statistics and physics. We usually want to solve multivariate problems to within an error threshold ε by algorithms that use finitely many function values or, more generally, finitely many continuous linear functionals. Let $n(\varepsilon, d)$ be the information complexity or shortly the complexity, denoting the minimal number of function values or continuous linear functionals that are needed to find an algorithm approximating the solution of a multivariate problem to within ε.

Many multivariate problems suffer from the *curse of dimensionality*. That is, $n(\varepsilon, d)$ is exponentially large in d. One of the goals of tractability is to determine under which conditions the curse of dimensionality is *not* present. Even more, we would like to have the complexity bounded by some non-exponential function of d and ε^{-1}. In particular, we have

- *weak* tractability if the complexity is *not* exponential in d or ε^{-1},
- *quasi-polynomial* tractability if the complexity is of order $\exp(\,t\,(1+\ln\, d)(1+\ln\, \varepsilon^{-1}))$,
- *polynomial* tractability if the complexity is of order $d^{\,q}\,\varepsilon^{-p}$,
- *strong* polynomial tractability if the complexity is of order ε^{-p}.

All bounds above hold for all d and all $\varepsilon \in (0,1)$ with the parameters t, q, p and the factors multiplying the corresponding complexity bounds independent of d and ε^{-1}.

The strong polynomial tractability is the most challenging property. If this holds then the complexity has a bound independent of d. One may think that this property may hold only for trivial problems. Luckily, the opposite is true.

The curse of dimensionality often holds for multivariate problems for which all variables and groups of variables play the same role. One way to vanquish the curse is to shrink the class of functions by introducing weights that monitor the influence of successive variables and groups of variables. For sufficiently fast decaying weights we not only vanquish the curse but even obtain strong polynomial tractability; a survey of such results may be found in [10, 11, 12].

The other way to vanquish the curse is by increasing the smoothness of functions with respect to successive variables. This approach was taken recently in [13]. It was done for

multivariate approximation defined over Korobov spaces in the worst case setting. The current paper can be viewed as a continuation of [13]. We consider multivariate approximation but now in the average case setting with the normalized error criterion. This error criterion is defined as follows. We first take the zero algorithm and find its average case error for multivariate approximation; this is called the initial error. The initial error tells us how the problem scales and what can be achieved without sampling the functions. The normalized error criterion means that we want to improve the initial error by a factor ε. We analyze algorithms that use arbitrary continuous linear functionals. We stress that the same results hold for algorithms that use only function values. This is due to general relations between these two classes of algorithms established in [5] and in Chapter 24 of [12].

In this paper we analyze two multivariate approximation problems defined for the Euler and Wiener integrated processes, whereas in [7] we consider average case approximation for general non-homogeneous tensor products. More precisely, here we take the space of continuous real functions defined on the d-dimensional unit cube $[0,1]^d$. We stress that d can be an arbitrary positive integer, however, our emphasis is on large d. We equip this space with a zero-mean Gaussian measure whose covariance kernel is denoted by K_d. We study two such kernels. The first one is $K_d = K_d^{\mathrm{E}}$ for the Euler integrated process, whereas the second one $K_d = K_d^{\mathrm{W}}$ is for the Wiener integrated process. These processes are precisely defined in the next section. Here we only mention that for both of them we know that almost surely the functions are r_k times continuously differentiable with respect to the k-th variable for $k = 1, 2, \dots, d$.

The information complexity is then denoted by $n^{\mathrm{E}}(\varepsilon, d)$ and $n^{\mathrm{W}}(\varepsilon, d)$ for the Euler and Wiener integrated processes, respectively. Obviously, it depends on the the smoothness parameters $\{r_k\}$. Our main goal in this paper is to find necessary and sufficient conditions in terms of $\{r_k\}$ such that the four notions of tractability are satisfied.

We now briefly describe the results obtained in this paper. For both processes we prove that weak tractability holds iff $\lim_{k\to\infty} r_k = \infty$. Otherwise, if $r = \lim_{k\to\infty} r_k < \infty$ then we have the curse of dimensionality. This means that if all $r_k \le r < \infty$ then both $n^{\mathrm{E}}(\varepsilon, d)$ and $n^{\mathrm{W}}(\varepsilon, d)$ depend exponentially on d and this holds for all $\varepsilon \in (0,1)$. Hence, the function $n^{\mathrm{x}}(\cdot, d)$ is discontinuous at 1. Indeed, $n^{\mathrm{x}}(1, d) = 0$ although for ε pathologically close to one $n^{\mathrm{x}}(\varepsilon, d)$ depends exponentially on d; here $\mathrm{x} \in \{\mathrm{E},\mathrm{W}\}$.

We stress that weak tractability does not depend on the rate of convergence of r_k to infinity. However, if we want to obtain other types of tractability we must require a certain convergence rate for the r_k, although the rate is different for the Euler and the Wiener case. For simplicity, let us consider

$$r_k = \lceil 1 + a \ln(1 + k \ln k) \rceil$$

for some positive number a. Then for the Euler case we have:

- $a < \frac{1}{2\,\ln 3}$ no quasi-polynomial tractability,
- $a = \frac{1}{2\,\ln 3}$ quasi-polynomial tractability but no polynomial tractability,
- $a > \frac{1}{2\,\ln 3}$ strong polynomial tractability.

For the Wiener case we have to assume much more since for r_k given above only weak tractability holds. For

$$r_k = \left\lceil k^s \, \ln^2(1+k) \right\rceil$$

we have

- $s < \frac{1}{2}$ no quasi-polynomial tractability,
- $s = \frac{1}{2}$ quasi-polynomial tractability but no polynomial tractability,
- $s > \frac{1}{2}$ strong polynomial tractability.

For general $\{r_k\}$, we prove that quasi-polynomial tractability holds iff

$$\begin{aligned}
\text{For the Euler case} \quad &: \quad \limsup_{d\to\infty} \frac{1}{\ln d} \sum_{k=1}^{d} (1+r_k)\, 3^{-2r_k} < \infty, \\
\text{For the Wiener case:} \quad &: \quad \limsup_{d\to\infty} \frac{1}{\ln d} \sum_{k=1}^{d} (1+r_k)^{-2} \, \max(1, \ln\, r_k) < \infty.
\end{aligned}$$

Furthermore, for both processes polynomial tractability is equivalent to strong polynomial tractability and holds iff

$$\begin{aligned}
\text{For the Euler case} \quad &: \quad a_{\mathrm{E}} := \liminf_{d\to\infty} \frac{r_k}{\ln\, k} > \frac{1}{2\,\ln 3}, \\
\text{For the Wiener case:} \quad &: \quad \liminf_{d\to\infty} \frac{r_k}{k^s} > 0 \qquad \text{for some} \quad s > \tfrac{1}{2}.
\end{aligned}$$

We also study the exponent $p^{\mathrm{str-avg-x}}$ of strong polynomial tractability which is defined as the infimum of p for which the complexity is of order ε^{-p}. For the Euler case we have

$$p^{\mathrm{str-avg-E}} = \max\left(\frac{2}{2r_1+1}, \frac{2}{2a_{\mathrm{E}}\,\ln 3 - 1} \right)$$

For the Wiener case and $r_k = k^s$ for some $s > \frac{1}{2}$ we have

$$\max\left(\frac{2}{2r_1+1}, \frac{2}{2s-1} \right) \le p^{\mathrm{str-avg-W}} \le \max\left(\frac{2}{2s-1}, 3 \right).$$

Hence, for $s \in (\frac{1}{2}, \frac{5}{6}]$ we know that

$$p^{\mathrm{str-avg-W}} = \frac{2}{2s-1}\,;$$

otherwise our bounds are too weak to provide the exact value of the exponent.

Our results solve a special case of Open Problem 11 in [10], where $r_{d,k} = r_k$, $k = 1, \dots, d$, and, with slightly modified proofs, they also solve Open Problem 10 in [10].

The Euler, Wiener and other univariate integrated processes can be characterized as follows. Consider

$$X^{\alpha}(t) := (-1)^{\alpha_1 + \cdots + \alpha_r} \underbrace{\int_{\alpha_r}^{t} \int_{\alpha_{r-1}}^{t_{r-1}} \dots \int_{\alpha_1}^{t_1} W(s)\, \mathrm{d}s \mathrm{d}t_1 \dots \mathrm{d}t_{r-1}}_{r \text{ times}}, \qquad 0 \le t \le 1,$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ is a multi-index with components $\alpha_i \in \{0, 1\}$ for $i = 1, 2, \dots, r$, $W(s)$ is the standard Wiener process for $0 \le s \le 1$, and $r \in \mathbb{N}$. Then $X^{(0,\dots,0)}$ is the integrated Wiener process and $X^{(1,0,1,0,1,\dots)}$ is the integrated Euler process. It is an open problem to consider the integrated processes resulting from the different values of the multi-index α and to compare the necessary and sufficient conditions on $\{r_k\}$ for weak, quasi-polynomial and polynomial tractability, respectively, with those obtained for the Euler and Wiener processes. In particular, it seems of interest to verify whether the Euler process is the easiest and the Wiener process is the most difficult among all of these 2^r processes.

The paper is organized as follows. In Section 2 we present the precise definitions of the average case approximation problem, the Euler and Wiener integrated processes and tractability notions. In Section 3 we present results for the Euler and in Section 4 for the Wiener integrated processes. The proofs of three theorems are presented in Sections 5 to 7.

2 Preliminaries

In this section we precisely define the Euler and Wiener processes, multivariate approximation in the average case setting, and we cite known results that will be needed for our analysis.

2.1 Euler and Wiener Processes

Let $F_d = C([0,1]^d)$ be the space of real continuous functions defined on $[0,1]^d$ with the max norm,

$$\|f\|_{F_d} = \max_{x \in [0,1]^d} |f(x)| \qquad \text{for all} \quad f \in F_d.$$

We equip the space F_d with a zero-mean Gaussian measure μ_d defined on Borel sets of F_d. The covariance kernel K_d related to μ_d is defined by

$$K_d(x,t) = \int_{F_d} f(x)\, f(t)\, \mu_d(\mathrm{d}f) \qquad \text{for all} \quad x,t \in [0,1]^d.$$

We refer to [6] for extensive theory of Gaussian measures in linear spaces and their covariance kernels.

By $\{r_k\}$ we mean a sequence of non-negative non-decreasing integers

$$0 \le r_1 \le r_2 \le \cdots \le r_d \le \cdots .$$

The Euler and Wiener integrated processes differ in the choice of the covariance kernel K_d. Our presentation of the Euler integrated processes is based on [1] and [3]. The Wiener integrated process is more standard and can be found in many books and papers.

- *Euler integrated process.* We now have $K_d = K_d^{\mathrm{E}}$ given by

  $$K_d^{\mathrm{E}}(x,y) = \prod_{k=1}^{d} K_{1,r_k}^{\mathrm{E}}(x_k,y_k) \quad \text{for all} \quad x,y \in [0,1]^d,$$

  where

  $$K_{1,r}^{\mathrm{E}}(x,y) = \int_{[0,1]^r} \min(x,s_1)\, \min(s_1,s_2)\, \cdots\, \min(s_r,y)\, \mathrm{d}s_1\, \mathrm{d}s_2 \cdots \mathrm{d}s_r$$

  for all $x,y \in [0,1]$. This kernel is equal to

  $$K_{1,r}^{\mathrm{E}}(x,y) = (-1)^{r+1} \frac{2^{2r}}{(2r+1)!} \Big(E_{2r+1}(\tfrac{1}{2}|x-y|) - E_{2r+1}(\tfrac{1}{2}(x+y)) \Big)$$

  for all $x,y \in [0,1]$. Here, E_n is the n-th degree Euler polynomial which can be defined as the coefficient of the generating function

  $$\frac{2\exp(x\,t)}{\exp(x)+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \qquad \text{for all} \quad x,t \in \mathbb{R}.$$

  In particular, we have $E_0 = 1$, $E_1(x) = x - \frac{1}{2}$ and $E_2(x) = x^2 - x$.

  The process is called *Euler* due to the fact that the covariance kernel is expressed by Euler polynomials.

- *Wiener integrated process.* We now have $K_d = K_d^{\mathrm{W}}$ given by

$$K_d^{\mathrm{W}}(x,y) = \prod_{k=1}^{d} K_{1,r_k}^{\mathrm{W}}(x_k, y_k) \quad \text{for all} \quad x, y \in [0,1]^d,$$

where

$$K_{1,r}^{\mathrm{W}}(x,y) = \int_0^{\min(x,y)} \frac{(x-u)^r}{r!} \, \frac{(y-u)^r}{r!} \, \mathrm{d}u = \int_0^1 \frac{(x-u)_+^r}{r!} \, \frac{(y-u)_+^r}{r!} \, \mathrm{d}u$$

for all $x, y \in [0,1]$ and with the standard notation $t_+ = \max(t, 0)$.

Let us stress that the univariate Euler and Wiener processes are close relatives since they emerge from very similar integration schemes. Indeed, let $W(t), t \in [0,1]$, be a standard Wiener process, i.e. a Gaussian random process with zero mean and covariance

$$K_{1,0}^{\mathrm{E}}(s,t) = K_{1,0}^{\mathrm{W}}(s,t) = \min(s,t).$$

Consider two sequences of integrated processes X_r, Y_r on $[0,1]$ defined by $X_0 = Y_0 = W$, and for $r = 0, 1, 2, \ldots$

$$\begin{aligned} X_{r+1}(t) &= \int_0^t X_r(s) \mathrm{d}s \\ Y_{r+1}(t) &= \int_{1-t}^1 Y_r(s) \mathrm{d}s, \end{aligned}$$

Then the covariance kernel of X_r is $K_{1,r}^{\mathrm{W}}$ while the covariance kernel of Y_r is $K_{1,r}^{\mathrm{E}}$. Clearly, X_r and Y_r have the same smoothness properties. That is why different tractability results are surprising.

On the other hand, there are some differences between the two processes. The Gaussian measure μ_d on F_d corresponding to the covariance kernel K_d^{E} or K_d^{W} is concentrated on functions that are almost surely r_k-times continuously differentiable with respect to the k-th variable for $k = 1, 2, \ldots, d$, and satisfy certain boundary conditions which are different for the Euler and Wiener cases.

For the Euler case, we have

$$\frac{\partial^{k_1 + k_2 + \cdots + k_d}}{\partial x_1^{k_1} \, \partial x_2^{k_2} \cdots \partial x_d^{k_d}} f(x) = 0 \tag{1}$$

if for some i we have $x_i = 0$ and k_i is even, or $x_i = 1$ and k_i is odd. Here, $k_i = 0, 1, \ldots, r_i$.

For the Wiener case, we have

$$\frac{\partial^{k_1+k_2+\cdots+k_d}}{\partial x_1^{k_1}\,\partial x_2^{k_2}\cdots\partial x_d^{k_d}}\,f(x)=0 \tag{2}$$

if one of the components of x is zero. As before, $k_i = 0, 1, \dots, r_i$.

To see the difference between (1) and (2) more explicitly, we take $d = 1$. Then for the Euler case for all $0 \le k \le r_1$ we have

$$f^{(k)}(0)=0 \quad \text{if } k \text{ is even} \qquad \text{and} \qquad f^{(k)}(1)=0 \;\; \text{if } k \text{ is odd},$$

whereas for the Wiener case we have

$$f^{(k)}(0)=0 \quad \text{for all } k \le r_1.$$

Finally, note that Nazarov and Nikitin studied in [8, 9] a slightly different version W_r^N of Euler integrated process. The processes W_r^N and W_r^{E} coincide for even r but $W_r^N(t) = W_r^{\mathrm{E}}(1-t)$ for odd r. The covariance spectra of both processes are the same but the boundary conditions are different. Since the spectra are the same, the tractability results for the Nazarov and Nikitin process are the same as for the Euler process.

2.2 Multivariate Approximation

Multivariate approximation is defined by the embedding $\mathrm{APP}_d : F_d \to L_2$ given by

$$\mathrm{APP}_d f = f \qquad \text{for all} \quad f \in F_d.$$

Here, $L_2 = L_2([0,1]^d)$ is the standard L_2 space with the norm

$$\|f\|_{L_2} = \left(\int_{[0,1]^d} f^{\,2}(t)\,\mathrm{d}t\right)^{1/2}.$$

We approximate functions f from F_d by algorithms A_n that use n function values or arbitrary continuous linear functionals. We only consider the case of arbitrary continuous functionals since it is known that the results are the same for function values, see [5] and Chapter 24 of [12]. In the average case setting, without essential loss of generality, see e.g., [15] as well as [10], we can restrict ourselves to linear algorithms A_n of the form

$$A_n(f) = \sum_{j=1}^{n} L_j(f)\,g_j \qquad \text{with} \qquad L_j \in F_d^*, \quad g_j \in L_2.$$

The average case error of A_n is defined as

$$e^{\mathrm{avg}}(A_n) = \left(\int_{F_d} \|\mathrm{APP}_d f - A_n(f)\|_{L_2}^2 \, \mu_d(\mathrm{d}f) \right)^{1/2},$$

where μ_d is a zero-mean Gaussian measure with a covariance kernel K_d and

$$\|\mathrm{APP}_d f - A_n(f)\|_{L_2}^2 = \int_{[0,1]^d} (f(t) - A_n(f)(t))^2 \, \mathrm{d}t.$$

Then $\nu_d = \mu_d \mathrm{APP}_d^{-1}$ is a zero-mean Gaussian measure defined on Borel sets of L_2. The covariance operator $C_{\nu_d} : L_2 \to L_2$ of ν_d is given by

$$C_{\nu_d} f = \int_{[0,1]^d} K_d(\cdot, t) \, f(t) \, \mathrm{d}t \qquad \text{for all} \quad f \in L_2.$$

The operator C_{ν_d} is a self-adjoint, nonnegative definite, and has finite trace. Let $(\lambda_{d,j}, \eta_{d,j})_{j=1,2,\dots}$ denote its eigenpairs

$$C_{\nu_d} \eta_{d,j} = \lambda_{d,j} \, \eta_{d,j} \qquad \text{with} \quad \lambda_{d,1} \geq \lambda_{d,2} \geq \cdots,$$

and

$$\sum_{j=1}^{\infty} \lambda_{d,j} = \int_{[0,1]^d} K_d(t,t) \, \mathrm{d}t < \infty.$$

For a given n, it is well known that the algorithm A_n that minimizes the average case error is of the form

$$A_n(f) = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{L_2} \, \eta_{d,j}, \tag{3}$$

and its average case error is

$$e^{\mathrm{avg}}(A_n) = \left(\sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{1/2}. \tag{4}$$

For $n = 0$ we obtain the zero algorithm $A_0 = 0$. Its average case error is called the initial error, and is given by the square-root of the trace of the operator C_{ν_d}, i.e., by (4) with $n = 0$.

We now define the average case information complexity $n(\varepsilon, d)$ as the minimal n for which there is an algorithm whose average case error reduces the initial error by a factor ε,

$$n(\varepsilon, d) = \min \left\{ n \;\middle|\; \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \sum_{j=1}^{\infty} \lambda_{d,j} \right\}. \tag{5}$$

From (5) it is clear that all notions of tractability depend only on the eigenvalues $\lambda_{d,j}$. Therefore the more we know about the eigenvalues $\lambda_{d,j}$ the more we can say about various notions of tractability.

2.3 Eigenvalues for the Euler and Wiener Integrated Processes

For both processes the corresponding covariance kernel is of product form. Therefore the eigenvalues for the d-variate case are products of the eigenvalues of the univariate cases which depend on the smoothness parameters r_k for $k = 1, 2, \dots, d$. That is, if we denote by $\lambda^{\mathrm{x}}_{d,j}$'s the eigenvalues of the Euler integrated process, x = E, or the eigenvalues of the Wiener integrated process, x = W, then

$$\{\lambda^{\mathrm{x}}_{d,j}\}_{j=1,2,\dots} = \left\{\lambda^{\mathrm{x}}_{j_1,r_1}\lambda^{\mathrm{x}}_{j_2,r_2}\dots\lambda^{\mathrm{x}}_{j_d,r_d}\right\}_{j_1,j_2,\dots,j_d=1,2\dots},$$

with the $\lambda^{\mathrm{x}}_{j_k,r_k}$'s denoting the eigenvalues of the univariate case with smoothness r_k.

For the Euler case, the $\lambda^{\mathrm{E}}_{j_k,r_k}$'s are the eigenvalues of the operator

$$(C^{\mathrm{E}}_{1,r_k} f)(x) = \int_0^1 K^{\mathrm{E}}_{1,r_k}(x,t)\, f(t)\, \mathrm{d}t.$$

By successive differentiation of this equation with respect to x and using the properties of the kernel K^{E}_{1,r_k}, it is easy to show that the eigenvalues of C^{E}_{1,r_k} satisfy the Sturm-Liouville problem

$$\lambda\, f^{(2r_k+2)}(x) = (-1)^{r_k+1} f(x) \qquad \text{for all} \quad x \in (0,1), \tag{6}$$

with the boundary conditions

$$f(t_0) = f'(t_1) = f''(t_2) = \dots = f^{(2r_k+1)}(t_{2r_k+1}) = 0,$$

where $t_i = 0$ for even i and $t_i = 1$ for odd i. For the Euler case, we know the eigenvalues exactly, see [1] and [3], and they are equal to

$$\lambda^{\mathrm{E}}_{j,r_k} = \left(\frac{1}{\pi(j-1/2)}\right)^{2r_k+2} \qquad \text{for} \quad j = 1, 2, \dots. \tag{7}$$

Note that the eigenvalues are well separated. In particular,

$$\frac{\lambda^{\mathrm{E}}_{2,r_k}}{\lambda^{\mathrm{E}}_{1,r_k}} = \frac{1}{3^{2r_k+2}}.$$

For the Wiener case, $\lambda^{\mathrm{W}}_{j,r_k}$'s are the eigenvalues of the operator

$$(C^{\mathrm{W}}_{1,r_k} f)(x) = \int_0^1 K^{\mathrm{W}}_{1,r_k}(x,t)\, f(t)\ \mathrm{d}t.$$

The eigenvalues $\lambda^{\mathrm{W}}_{j,r_k}$ also satisfy the Sturm-Liouville problem (6) but with different boundary conditions

$$f(0) = f'(0) = \cdots = f^{(r_k)}(0) = f^{(r_k+1)}(1) = f^{(r_k+2)}(1) = \cdots = f^{(2r_k+1)}(1) = 0.$$

The eigenvalues $\lambda^{\mathrm{W}}_{j,r_k}$ are *not* exactly known. It is known [3] that they have the same asymptotic behavior as in (7),

$$\lambda^{\mathrm{W}}_{j,r_k} = \left(\frac{1}{\pi(j-1/2)}\right)^{2r_k+2} + \mathcal{O}\left(j^{-(2r_k+3)}\right) \qquad \text{as } \ j \to \infty. \tag{8}$$

For tractability studies the asymptotic behavior is not enough and the two largest eigenvalues play an essential role. That is why we will prove that

$$\begin{aligned} \lambda^{\mathrm{W}}_{1,r_k} &= \frac{1}{(r_k!)^2}\left(\frac{1}{(2r_k+2)(2r_k+1)} + \mathcal{O}(r_k^{-4})\right), \\ \lambda^{\mathrm{W}}_{2,r_k} &= \Theta\left(\frac{1}{(r_k!)^2\, r_k^4}\right), \end{aligned}$$

where the factors in the big $\mathcal{O}$ and Θ notations do not depend on r_k.

Note that the largest eigenvalues for the Euler case go to zero exponentially fast with r_k, whereas for the Wiener case they go to zero super exponentially fast due to the presence of factorials. However, the ratio of the two largest eigenvalues for the Wiener case,

$$\frac{\lambda^{\mathrm{W}}_{2,r_k}}{\lambda^{\mathrm{W}}_{1,r_k}} = \Theta(r_k^{-2}),$$

is much larger than that for the Euler case.

2.4 Tractability

We present the precise definitions of four notions of tractability. Let $n(\varepsilon,d)$ denote the average case information complexity defined in (5), and let $\mathrm{APP} = \{\mathrm{APP}_d\}_{d=1,2,\ldots}$ denote the sequence of multivariate approximation problems. We say that

- APP is *weakly tractable* iff
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon,d)}{\varepsilon^{-1}+d} = 0,$$
with the convention that $\ln 0 = 0$.

- APP is *quasi-polynomially tractable* iff there are positive numbers C and t such that

$$n(\varepsilon, d) \le C\, \exp\left(t\,(1+\ln\, d)\,(1+\ln\, \varepsilon^{-1}\right) \qquad \text{for all} \quad d=1,2,\dots, \quad \varepsilon\in(0,1).$$

- APP is *polynomially tractable* iff there are non-negative numbers C, q and p such that

$$n(\varepsilon, d) \le C\, d^{\,q}\, \varepsilon^{-p} \qquad \text{for all} \quad d=1,2,\dots, \quad \varepsilon\in(0,1).$$

- APP is *strongly polynomially tractable* iff there are positive numbers C and p such that

$$n(\varepsilon, d) \le C\, \varepsilon^{-p} \qquad \text{for all} \quad d=1,2,\dots, \quad \varepsilon\in(0,1).$$

  The infimum of p satisfying the last bound is called the exponent of strong polynomial tractability and denoted by $p^{\rm str-avg}$. For the Euler and Wiener case, we use the notation $p^{\rm str-avg-x}$ with $\mathrm{x}\in\{\mathrm{E},\mathrm{W}\}$.

Tractability can be fully characterized in terms of the eigenvalues $\lambda_{d,j}$. Necessary and sufficient conditions on weak, quasi-polynomial, polynomial and strong polynomial tractability can be found in Chapter 6 of [10] and Chapter 24 of [12] as well as in [7] for non-homogeneous tensor products. For the Euler and Wiener integrated processes we need such conditions that are based on the sums of some power of the eigenvalues $\lambda_{d,j}$. We will cite these conditions when they are needed for specific tractability results.

3 Euler Integrated Process

We now analyze the Euler integrated process for which the eigenvalues in the univariate cases are given by (7). Our aim is to express tractability conditions in terms of the smoothness parameters $\{r_k\}$.

Theorem 1 *Consider the approximation problem* APP *for the Euler integrated process.*

- APP *is weakly tractable iff*

$$\lim_{k\to\infty} r_k = \infty. \tag{9}$$

  Furthermore, if (9) *does not hold then we have the curse of dimensionality since* $n^{\rm E}(\varepsilon, d)$ *depends exponentially on* d *for each* $\varepsilon < 1$.

- APP *is quasi-polynomially tractable iff*

$$\sup_{d\in\mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} (1+r_k)\, 3^{-2r_k} < \infty, \tag{10}$$

  where $\ln_+ d = \max(1, \ln\, d)$.

- APP *is polynomially tractable iff* APP *is strongly polynomially tractable iff*

$$\sum_{k=1}^{\infty} 3^{-2\,\tau\, r_k} < \infty \qquad \textit{for some} \ \ \tau \in (0,1)$$

or equivalently iff

$$a_{E} := \liminf_{k\to\infty} \ \frac{r_k}{\ln\, k} > \frac{1}{2\ln\, 3}.$$

If so, then the exponent[1] *of strong polynomial tractability is*

$$p^{\mathrm{str-avg-E}} = \max\left(\frac{2}{2r_1+1}, \frac{2}{2a_{E}\,\ln\,3-1}\right).$$

We briefly comment on Theorem 1. First of all, we stress that polynomial and strong tractability are equivalent. That is, these two notions coincide for the Euler integrated process: in this case a "weaker"" property of polynomial tractability implies a "stronger" property of strong polynomial tractability. Weak tractability requires that the smoothness parameters r_k go to infinity, however, the speed of convergence is irrelevant. To obtain at least quasi-polynomial tractability, we need to assume that r_k increases at least as $a_{\mathrm{E}}\,\ln\,k$ with $a_{\mathrm{E}} > 1/(2\ln\,3)$. Indeed, assume for simplicity that

$$a_{\mathrm{E}} := \lim_{k\to\infty} \frac{r_k}{\ln\,k}.$$

exists. If $a_{\mathrm{E}} < 1/(2\,\ln\,3)$ then for any positive $\beta < 1 - 2\,a_{\mathrm{E}}\,\ln\,3$ we have

$$n^{\mathrm{E}}(\varepsilon,d) \ge c_1(\beta)\,(1-\varepsilon^2)\,\exp\left((c_2(\beta)\,d^{\beta}\right) \tag{11}$$

for some positive functions c_1 and c_2 of β. Note that (11) contradicts quasi-polynomial tractability. The proof of (11) goes like follows. We will show later that

$$n^{\mathrm{E}}(\varepsilon,d) \ge (1-\varepsilon^2)\prod_{k=1}^{d}\left(1+3^{-2(r_k+1)}\right).$$

Then each factor $1+3^{-2(r_k+1)}$ for large j can be estimated from below by $\exp(-c(\beta)k^{-1+\beta})$. From this we easily obtain (11).

[1] It may happen that $a_{\mathrm{E}} = \infty$. Then the second term in the maximum defining $p^{\mathrm{str-avg-E}}$ is zero.

If $a_{\mathrm{E}} = 1/(2 \ln 3)$ then we can have quasi-polynomial tractability as illustrated by an example of $\{r_k\}$ in the introduction. Furthermore, for this example we do not have polynomial tractability. However, it may also happen that for $a_{\mathrm{E}} = 1/(2 \ln 3)$ we do not have quasi-polynomial tractability. For example, this is the case when

$$r_k = \left\lceil \frac{\ln_+ k}{2 \ln 3} \right\rceil,$$

which can be checked directly from (10).

On the other hand, if $a_{\mathrm{E}} > 1/(2 \ln 3)$ then we obtain strong polynomial tractability. This shows that there is a "thin" zone of $\{r_k\}$ that separates quasi-polynomial and strong polynomial tractabilities.

We now comment on the exponent of strong polynomial tractability. Note that for $a_{\mathrm{E}} \geq (r_1 + 1)/\ln 3$ we have

$$p^{\mathrm{str-avg-E}} = \frac{2}{2r_1 + 1}.$$

In this case, the result is especially pleasing hence the complexity for any d is roughly bounded by the complexity for the univariate case. Furthermore, this happens for all r_k's that tend to infinity faster than $\ln k$. On the other hand, if $a_{\mathrm{E}} \in (1/(2 \ln 3)), 2(r_1 + 1)/(2 \ln 3))$ then we have

$$p^{\mathrm{str-avg-E}} = \frac{2}{2a_{\mathrm{E}} \ln 3 - 1},$$

and $p^{\mathrm{str-avg-E}}$ can be arbitrarily large when a_{E} is close to $1/(2 \ln 3)$.

4 Wiener Integrated Process

We now turn to the Wiener integrated process for which the eigenvalues for the univariate cases $\lambda^{\mathrm{W}}_{j,r_k}$ are only known asymptotically, see (8). To express tractability conditions in terms of the smoothness parameters $\{r_k\}$ we will need to prove the behavior of the two largest eigenvalues for large r_k.

Theorem 2 *Consider the univariate Wiener process with the smoothness parameter r, and let $\lambda^{W}_{j,r}$'s denote the eigenvalues of the covariance operator $C^{W}_{1,r}$. Then*

$$
\begin{aligned}
\lambda_{1,r}^{W} &= \frac{1}{(r!)^2}\left(\frac{1}{(2r+2)\,(2r+1)} + \mathcal{O}(r^{-4})\right), \\
\lambda_{2,r}^{W} &= \Theta\left(\frac{1}{(r!)^2\, r^4}\right), \\
\sup_{\tau\in[\tau_0,1]} \frac{\sum_{j=3}^{\infty}\left[\lambda_{j,r}^{W}\right]^{\tau}}{\left[\lambda_{2,r}^{W}\right]^{\tau}} &= \mathcal{O}(r^{-h}) \qquad \textit{for some } h>0 \quad \textit{and for all} \quad \tau_0 \in (\tfrac{3}{5}, 1].
\end{aligned}
$$

Observe that the two largest eigenvalues for the Wiener case are much smaller than for the Euler case. On the other hand, their ratio for the Wiener case is much larger than for the Euler case. Therefore, the sequences $\{\lambda_{j,r}^{\mathrm{W}}\}$ and $\{\lambda_{j,r}^{\mathrm{E}}\}$ are quite different although they have the same asymptotic behavior.

The uniform convergence in the last assertion of Theorem 2 at the neighborhood of $\tau = 1$ is needed when we deal with quasi-polynomial tractability. The convergence for a specific τ is needed for strong polynomial and polynomial tractability. The lower bound $\frac{3}{5}$ for τ_0 is surely not sharp. A possible improvement of this lower bound would improve the exponent of strong polynomial tractability.

Based on the estimates presented in Theorem 2 we will be able to express tractability conditions for the Wiener case in terms of $\{r_k\}$.

Theorem 3 *Consider the approximation problem* APP *for the Wiener integrated process.*

- APP *is weakly tractable iff*
$$\lim_{k\to\infty} r_k = \infty. \tag{12}$$
Furthermore, if (12) *does not hold then we have the curse of dimensionality since* $n^{W}(\varepsilon, d)$ *depends exponentially on* d *for each* $\varepsilon < 1$.

- APP *is quasi-polynomially tractable iff*
$$\sup_{d\in\mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} (1+r_k)^{-2} \ln_+ r_k < \infty, \tag{13}$$
where, we use $\ln_+ x = \max(1, \ln x)$ *for* $x>0$, *and* $\ln_+ 0 = 1$.

- APP *is polynomially tractable iff* APP *is strongly polynomially tractable iff*
$$\liminf_{k\to\infty} \frac{r_k}{k^s} > 0 \quad \textit{for some} \quad s > \tfrac{1}{2}.$$

We briefly comment on Theorem 3. As for the Euler case, strong polynomial and polynomial tractability are equivalent, and weak tractability holds under the same condition $\lim_k r_k = \infty$. That ends the similarity between the Wiener and Euler cases since the conditions on quasi-polynomial and polynomial tractability are quite different. For the Wiener case, we must assume that r_k's go to infinity at least as fast as k^{-s} for some $s > \frac{1}{2}$. However, the zone between quasi-polynomial and polynomial tractabilities is again thin, as for the Euler case.

It is worth to add that quasi-polynomial tractability plays a much more important role in the worst case setting. The difference with the average case setting is due to the fact that even for the constant sequence $r_k = \text{const} > 0$ we have quasi-polynomial tractability in the worst case setting as shown in [4].

We now discuss the exponent of strong tractability which is not addressed in Theorem 3. For simplicity, let us assume that for some $s > \frac{1}{2}$ we have

$$r_k = k^s \quad \text{for all} \quad k \in \mathbb{N}.$$

Then we have strong polynomial tractability and the exponent $p^{\text{str-avg-W}}$ is given in (16) as the infimum of $2\tau/(1-\tau)$ for τ from $(0,1)$ which satisfies condition (15) below with $q = 0$. From the proof of Theorem 3 we know that $\tau > 1/(2r_1 + 2)$. Furthermore, (48) implies that $\tau > 1/(2s)$. These two estimates yield lower bounds on the exponent. On the other hand, our proof of strong polynomial tractability is valid only for $\tau > \frac{3}{5}$, and this effects an upper bound on the exponent. Hence,

$$\max\left(\frac{2}{2r_1+1}, \frac{2}{2s-1}\right) \le p^{\text{str-avg-W}} \le \max\left(\frac{2}{2s-1}, 3\right).$$

We stress that only for $s \in (\frac{1}{2}, \frac{5}{6}]$ we know the exponent exactly, $p^{\text{str-avg-W}} = \frac{2}{2s-1}$. Note that $p^{\text{str-avg-W}}$ can be arbitrarily large if s is close to $\frac{1}{2}$.

For $s > \frac{5}{6}$, our bounds on the eigenvalues $\lambda^{\text{W}}_{j,r_k}$ are too weak to get the exact value of the exponent but sufficient to deduce strong polynomial tractability.

5 Proof of Theorem 1

It is convenient to deal first with polynomial tractability. Let PT stand for polynomial tractability and SPT for strong polynomial tractability. To prove this point of Theorem 1 it is enough to show that

$$a_{\text{E}} > \frac{1}{2\,\ln 3} \;\Rightarrow\; \sum_{k=1}^{\infty} 3^{-2\tau\, r_k} < \infty \;\Rightarrow\; \text{SPT} \;\Rightarrow\; \text{PT} \Rightarrow\; a_{\text{E}} > \frac{1}{2\,\ln 3}\,. \tag{14}$$

The first claim, $a_{\mathrm{E}} > 1/(2\ln 3) \;\Rightarrow S_\tau := \sum_{k=1}^\infty 3^{-2\tau\, r_k} < \infty$ for some $\tau \in (0,1)$, is an easy calculus exercise. Indeed, let $a_{\mathrm{E}} > 1/(2\ln 3)$. Then for some $\delta > 0$ and all k large enough we have $\frac{r_k}{\ln k} > \frac{1+\delta}{2\ln 3}$, hence $3^{-2\tau r_k} < k^{-(1+\delta)\tau}$ and $S_\tau < \infty$ whenever $\frac{1}{1+\delta} < \tau < 1$.

Recall now the polynomial tractability criteria. We know from Chapter 6 of [10] that APP is polynomially tractable iff there exist $q \ge 0$ and $\tau \in (0,1)$ such that

$$C := \sup_{d\in\mathbb{N}} \frac{\left(\sum_{j=1}^\infty \lambda_{d,j}^\tau\right)^{1/\tau}}{\sum_{j=1}^\infty \lambda_{d,j}}\; d^{-q} < \infty. \tag{15}$$

If so then

$$n(\varepsilon,d) \le \left(\left(\frac{\tau\, C}{1-\tau}\right)^{\tau/(1-\tau)} + 1\right) d^{\,q\,\tau/(1-\tau)}\, \varepsilon^{-2\tau/(1-\tau)}$$

for all $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$.

Furthermore, APP is strongly polynomially tractable iff (15) holds with $q = 0$. The exponent of strong polynomial tractability is

$$p^{\mathrm{str-avg}} = \inf\left\{ \frac{2\tau}{1-\tau} \;\middle|\; \tau \text{ satisfies (15) with } q = 0 \right\}. \tag{16}$$

Motivated by condition (15) and based on the explicit knowledge of the univariate eigenvalues for the Euler integrated process (7), we take $\tau \in (0,1)$ and obtain

$$\begin{aligned}
\frac{\left(\sum_{j=1}^\infty \lambda_{d,j}^\tau\right)^{1/\tau}}{\sum_{j=1}^\infty \lambda_{d,j}} &= \prod_{k=1}^d \frac{\left(\sum_{j=1}^\infty \left(\lambda_{j,r_k}^{\mathrm{E}}\right)^\tau\right)^{1/\tau}}{\sum_{j=1}^\infty \lambda_{j,r_k}^{\mathrm{E}}} \\
&= \prod_{k=1}^d \frac{\left(\sum_{j=1}^\infty (2j-1)^{-(2r_k+2)\tau}\right)^{1/\tau}}{\sum_{j=1}^\infty (2j-1)^{-(2r_k+2)}} \\
&= \prod_{k=1}^d \frac{\left(1 + \sum_{j=2}^\infty (2j-1)^{-2\tau(r_k+1)}\right)^{1/\tau}}{1 + \sum_{j=2}^\infty (2j-1)^{-2(r_k+1)}}.
\end{aligned}$$

Since $r_k \ge r_1$, note that the expression above is finite for all $\tau \in (1/(2r_1+2), 1)$. Furthermore for such τ we have

$$3^{-2\tau(r_k+1)} \le \sum_{j=2}^\infty (2j-1)^{-2\tau(r_k+1)} \le 3^{-2\tau(r_k+1)} + \sum_{j=5}^\infty j^{-2\tau(r_k+1)},$$

and

$$\sum_{j=5}^{\infty} j^{-2\tau(r_k+1)} \leq \int_4^{\infty} x^{-2\tau(r_k+1)}\,\mathrm{d}x = \frac{4^{1-2\tau(r_k+1)}}{2\tau(r_k+1)-1} \leq \frac{3}{2\tau(r_1+1)-1}\,3^{-2\tau(r_k+1)}.$$

Therefore

$$\frac{\left(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty}\lambda_{d,j}} = \prod_{k=1}^{d}\frac{\left(1+a_k 3^{-2\tau(r_k+1)}\right)^{1/\tau}}{1+b_k 3^{-2(r_k+1)}}, \tag{17}$$

where $a_k \geq b_k$ and they are uniformly bounded,

$$1 \leq a_k \leq \frac{2\tau(r_1+1)+2}{2\tau(r_1+1)-1} \qquad \text{and} \qquad 1 \leq b_k \leq \frac{2r_1+4}{2r_1+1}. \tag{18}$$

Assume now that $S_\tau < \infty$ for some $\tau < 1$. By using (17) and (18) we obtain

$$\begin{aligned}\sup_d \frac{\left(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty}\lambda_{d,j}} &\leq \prod_{k=1}^{\infty}\left(1+a_k 3^{-2\tau(r_k+1)}\right)^{1/\tau} \\ &\leq \exp\left(\tau^{-1}\sup_k a_k \sum_{k=1}^{\infty} 3^{-2\tau(r_k+1)}\right) \leq \exp\left(\tau^{-1}\sup_k a_k\; S_\tau\right) < \infty.\end{aligned}$$

Hence, the criterion (15) is verified with $q = 0$, and we conclude that $S_\tau < \infty \Rightarrow$ SPT.

Implication SPT $\Rightarrow$ PT is trivial.

Assume now that PT holds. By (15) and (17) this implies that

$$\prod_{k=1}^{d}\frac{\left(1+a_k 3^{-2\tau(r_k+1)}\right)^{1/\tau}}{1+b_k 3^{-2(r_k+1)}} < C\,d^{\,q}$$

for some $C, q \geq 0$ and $\tau \in (0,1)$. Moreover, is easy to check that

$$\frac{\left(1+a_k 3^{-2\tau(r_k+1)}\right)^{1/\tau}}{1+b_k 3^{-2(r_k+1)}} \geq 1 + c_k 3^{-2\tau(r_k+1)}$$

for $c_k \geq a_k(1-3^{-2(r_k+1)(1-\tau)})/(1+b_k 3^{-2(r_k+1)}) = \Omega(1)$. Taking logarithms we conclude that

$$M := \sup_d \;\frac{1}{\ln_+ d}\;\sum_{k=1}^{d} 3^{-2\tau(r_k+1)} < \infty.$$

The sum with respect to k can be lower bounded by $d \cdot 3^{-2\tau(r_d+1)}$, as done at the beginning of the proof, and we obtain $d \cdot 3^{-2\tau(r_d+1)} \le M \ln_+ d$, which is equivalent to

$$\frac{r_d+1}{\ln d} \ge \frac{1 - \frac{\ln\ln_+ d - \ln M}{\ln d}}{\tau \cdot 2\ln 3},$$

and implies that $a_{\mathrm{E}} \ge 1/(2\tau \ln 3) > 1/(2 \ln 3)$, as claimed. The equivalence of all statements in (14) is therefore verified.

We now consider the exponent $p^{\mathrm{str-avg-E}}$. Assume now that $a_{\mathrm{E}} > \frac{1}{2\ln 3}$. Then, as already shown, $\sum_{k=1}^{\infty} 3^{-2\tau(r_k+1)} < \infty$ for all $\tau > \frac{1}{2a_{\mathrm{E}}\ln 3}$ and (15) holds with $q = 0$ if $\tau > \frac{1}{2r_1+2}$. Hence, we obtain strong polynomial tractability. Furthermore, τ can be taken in the limit as $\tau_* := \max\left(\frac{1}{2r_1+2}, \frac{1}{2a_{\mathrm{E}}\ln 3}\right)$, and (16) yields that the exponent of strong polynomial tractability is at most

$$p_* := \frac{2\tau_*}{1-\tau_*} = \max\left(\frac{2}{2r_1+1}, \frac{2}{2a_{\mathrm{E}}\ln 3 - 1}\right).$$

Conversely, assume that strong polynomial tractability holds. Then

$$\prod_{k=1}^{\infty} \frac{\left(1 + a_k 3^{-2\tau(r_k+1)}\right)^{1/\tau}}{1 + b_k 3^{-2(r_k+1)}} < \infty$$

for some $\tau \in (0,1)$. Clearly, we must take $\tau > 1/(2r_1+2)$ and $\tau > 1/(2a_{\mathrm{E}} \ln 3)$. This implies that the exponent is at least p_*. This completes the part of the proof related to polynomial and strong polynomial tractability.

We now turn to weak tractability. We know from [7] that APP is weakly tractable if there exists $\tau \in (0,1)$ such that

$$\lim_{d\to\infty} \frac{1}{d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda^{\mathrm{E}}_{j,r_k}}{\lambda^{\mathrm{E}}_{1,r_k}}\right)^{\tau} = 0. \tag{19}$$

In our case, we have

$$\frac{\lambda^{\mathrm{E}}_{j,r_k}}{\lambda^{\mathrm{E}}_{1,r_k}} = (2j-1)^{-2(r_k+1)}.$$

As before, for $\tau \in (\frac{1}{2}, 1)$ we have

$$\sum_{j=2}^{\infty} (2j-1)^{-2\tau(r_k+1)} \le \frac{2\tau(r_k+1)+2}{2\tau(r_k+1)-1}\, 3^{-2\tau(r_k+1)} \le \frac{2(1+\tau)}{2\tau-1}\, 3^{-2\tau(r_k+1)}.$$

Assume that $\lim_{k\to\infty} r_k = \infty$. Then for an arbitrarily large M there is an integer k_M such that $r_k \geq M$ for all $k \geq k_M$. Hence, for $d \geq k_M$ we have

$$\frac{1}{d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda^{\mathrm{E}}_{j,r_k}}{\lambda^{\mathrm{E}}_{1,r_k}} \right)^{\tau} \leq \frac{2(1+\tau)}{2\tau - 1} \left(\frac{k_M}{d} + 3^{-2\tau(M+1)} \right),$$

and we obtain (19) by letting first d, and then M go to infinity.

On the other hand, if $r = \lim_{k\to\infty} r_k < \infty$ then there is an integer k_0 such that $r_k = r$ for all $k \geq k_0$, and the limit in (19) is not zero. In this case, we prove that $n = n^{\mathrm{E}}(\varepsilon, d)$ is an exponential function of d and therefore weak tractability does not hold. Indeed, we have

$$\sum_{j=1}^{\infty} \lambda_{d,j} - n\lambda_{d,1} \leq \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \sum_{j=1}^{\infty} \lambda_{d,j},$$

and therefore

$$\begin{aligned} n \;&\geq\; (1-\varepsilon^2) \sum_{j=1}^{\infty} \frac{\lambda_{d,j}}{\lambda_{d,1}} = (1-\varepsilon^2) \left(\prod_{k=1}^{k_0-1} \sum_{j=1}^{\infty} \frac{\lambda^{\mathrm{E}}_{j,r_k}}{\lambda^{\mathrm{E}}_{1,r_k}} \right) \left(1 + \sum_{j=2}^{\infty} \frac{\lambda^{\mathrm{E}}_{j,r}}{\lambda^{\mathrm{E}}_{1,r}} \right)^{d-k_0+1} \\ &\geq\; (1-\varepsilon^2) \left(1 + \sum_{j=2}^{\infty} \frac{\lambda^{\mathrm{E}}_{j,r}}{\lambda^{\mathrm{E}}_{1,r}} \right)^{d-k_0+1}. \end{aligned}$$

This bound is an exponential function of d. It contradicts weak tractability and completes the part of the proof related to this notion.

We finally consider quasi-polynomial tractability. We know from [7] that APP is quasi-polynomially tractable iff there exists a positive δ such that

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\delta/\ln_+ d}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\delta/\ln_+ d}} < \infty, \tag{20}$$

where $\ln_+ d = \max(1, \ln\, d)$.

Sufficiency. We first prove that (10) implies (20) with $\delta = \frac{1}{2}$. Let

$$\lambda(j,k) = (2j-1)^{-2(r_k+1)}.$$

We have

$$\sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\frac{1}{2\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\frac{1}{2\ln_+ d}}} = \sup_{d\in\mathbb{N}} \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda(j,k)\right)^{1-\frac{1}{2\ln_+ d}}}.$$

We split the last product into two products

$$\Pi_1(d) := \prod_{k=1}^{d} \left(\sum_{j=1}^{\infty} \lambda(j,k) \right)^{\frac{1}{2 \ln_+ d}}$$

and

$$\Pi_2(d) := \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}}}{\sum_{j=1}^{\infty} \lambda(j,k)}.$$

In what follows we use C to denote a positive number which is independent of d and $\{r_k\}$, and whose value may change for successive estimates. For $\Pi_1(d)$ we simply have

$$\begin{aligned} \Pi_1(d) &= \prod_{k=1}^{d} \left(1 + \sum_{j=2}^{\infty} \lambda(j,k) \right)^{\frac{1}{2\ln_+ d}} \leq \exp\left(\frac{1}{2\ln_+ d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \lambda(j,k) \right) \\ &\leq \exp\left(\frac{C}{\ln_+ d} \sum_{k=1}^{d} \lambda(2,k) \right) = \exp\left(\frac{C}{\ln_+ d} \sum_{k=1}^{d} 3^{-2(r_k+1)} \right). \end{aligned}$$

Clearly, (10) implies that $\sup_{d\in\mathbb{N}} \Pi_1(d) < \infty$.

We now turn to the product $\Pi_2(d)$. We estimate each of its factors by

$$\begin{aligned} \frac{\sum_{j=1}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}}}{\sum_{k=1}^{\infty} \lambda(j,k)} &\leq \frac{1 + \lambda(2,k)^{1-\frac{1}{2\ln_+ d}} + \sum_{j=3}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}}}{1+\lambda(2,k)} \\ &\leq \frac{1+\lambda(2,k)^{1-\frac{1}{2\ln_+ d}}}{1+\lambda(2,k)} + \sum_{j=3}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}}. \end{aligned} \tag{21}$$

Note that if $|\ln \lambda(2,k)| \leq 3 \ln_+ d$, then

$$\begin{aligned} \frac{1+\lambda(2,k)^{1-\frac{1}{2\ln_+ d}}}{1+\lambda(2,k)} &= \frac{1+\lambda(2,k)\exp\left(\frac{-\ln\lambda(2,k)}{2\ln_+ d}\right)}{1+\lambda(2,k)} \\ &\leq \frac{1+\lambda(2,k)\left(1+\frac{C|\ln\lambda(2,k)|}{\ln_+ d}\right)}{1+\lambda(2,k)} \\ &\leq 1 + \frac{C\lambda(2,k)|\ln\lambda(2,k)|}{\ln_+ d}, \end{aligned}$$

while if $|\ln \lambda(2,k)| \geq 3 \ln_+ d$, then

$$\frac{1+\lambda(2,k)^{1-\frac{1}{2\ln_+ d}}}{1+\lambda(2,k)} \leq 1+\lambda(2,k)^{1-\frac{1}{2\ln_+ d}} \leq 1+\lambda(2,k)^{1/2} \leq 1+d^{-3/2}.$$

Thus, in any case

$$\frac{1+\lambda(2,k)^{1-\frac{1}{2\ln_+ d}}}{1+\lambda(2,k)} \leq 1+d^{-3/2}+\frac{C\lambda(2,k)|\ln \lambda(2,k)|}{\ln_+ d}. \tag{22}$$

Next, we have

$$\sum_{j=3}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}} \leq C\lambda(3,k)^{1-\frac{1}{2\ln_+ d}} = C\lambda(2,k)^{\frac{\ln 5}{\ln 3}(1-\frac{1}{2\ln_+ d})}. \tag{23}$$

We now show that (10) implies that $\lambda(2,k) = 3^{-2(r_k+1)} \leq C/k$. First of all note that (10) implies that $\lim_k r_k = \infty$, so that only finitely many initial r_k may be zero. Assume that d is so large that $r_d \geq 1$ and $d \geq 3$. Since $(1+r_k)3^{-2r_k}$ is non-increasing, we have

$$r_d 3^{-2r_d} \leq \frac{1}{d}\sum_{k=1}^{d}(1+r_k)3^{-2r_k} \leq \frac{C\ \ln d}{d},$$

so that $3^{2r_d} \geq 3^{2r_d}/r_d \geq d/(C\ \ln d)$ and

$$r_d \geq \frac{\ln d - \ln(C\ \ln d)}{2\ \ln 3} \geq C_1\ \ln d.$$

Hence,

$$\lambda(2,d) = 3^{-2(r_d+1)} \leq \frac{r_d 3^{-2r_d}}{r_d} \leq \frac{C\ \ln d}{r_d\, d} \leq \frac{C}{C_1\, d}$$

as claimed. By enlarging the constant, we obtain the same inequality for *all* d. For $k \leq d$, we then have by (23)

$$\sum_{j=3}^{\infty} \lambda(j,k)^{1-\frac{1}{2\ln_+ d}} \leq C\, k^{-\frac{\ln(5)}{\ln(3)}(1-\frac{1}{2\ln_+ d})} \leq C\, k^{-\frac{\ln(5)}{\ln(3)}} \tag{24}$$

Using $1+x \leq \exp(x)$, from (21), (22), and (24), we obtain

$$\frac{\sum_{j=1}^{\infty}\lambda(j,k)^{1-\frac{1}{2\ln_+ d}}}{\sum_{j=1}^{\infty}\lambda(j,k)} \leq \exp\left(d^{-3/2}+\frac{C\lambda(2,k)|\ln \lambda(2,k)|}{\ln_+ d}+Ck^{-\frac{\ln 5}{\ln 3}}\right).$$

Then it follows that

$$\begin{aligned}\Pi_2(d) &\le \exp\left(\sum_{k=1}^{d}\left(d^{-3/2}+\frac{C\lambda(2,k)|\ln\lambda(2,k)|}{\ln_+ d}+Ck^{-\frac{\ln 5}{\ln 3}}\right)\right)\\ &\le \exp\left(\sum_{k=1}^{d}\left(d^{-3/2}+\frac{C3^{-2r_k}(r_k+1)}{\ln_+ d}+Ck^{-\frac{\ln 5}{\ln 3}}\right)\right),\end{aligned}$$

and (10) implies that $\sup_{d\in\mathbb{N}}\Pi_2(d)<\infty$. Therefore,

$$\sup_{d\in\mathbb{N}}\Pi_1(d)\,\Pi_2(d)\le\sup_{d\in\mathbb{N}}\Pi_1(d)\quad\sup_{d\in\mathbb{N}}\Pi_2(d)<\infty,$$

the required property (20) is verified, so that the quasi-polynomial tractability is proved.

Necessity. Assume now that quasi-polynomial tractability holds. We prove in [7] that quasi-polynomial tractability implies

$$\sup_{d\in\mathbb{N}}\frac{1}{\ln_+ d}\sum_{k=1}^{d}\sum_{j=1}^{\infty}\frac{\lambda(j,k)}{\Lambda(k)}\ln\left(\frac{\Lambda(k)}{\lambda(j,k)}\right)<\infty, \tag{25}$$

where $\Lambda(k)=\sum_{j=1}^{\infty}\lambda(j,k)$. Clearly, $\Lambda(k)/\lambda(j,k)>1$ so that all terms in the sums over j are positive. We simplify the last condition by omitting all terms for $j\neq 2$, and obtain

$$\sup_{d\geq\mathbb{N}}\frac{1}{\ln_+ d}\sum_{k=1}^{d}\frac{\lambda(2,k)}{\Lambda(k)}\ln\left(\frac{\Lambda(k)}{\lambda(2,k)}\right)<\infty. \tag{26}$$

Next, since $\Lambda(k)>1$ we can also omit $\ln\Lambda(k)$ and obtain

$$\sup_{d\in\mathbb{N}}\frac{1}{\ln_+ d}\sum_{k=1}^{d}\frac{\lambda(2,k)}{\Lambda(k)}\ln\left(\frac{1}{\lambda(2,k)}\right)<\infty.$$

Furthermore, since $\{\Lambda(k)\}$ is non-increasing, we have

$$\sup_{d\in\mathbb{N}}\frac{1}{\ln_+ d}\sum_{k=1}^{d}\lambda(2,k)\ln\left(\frac{1}{\lambda(2,k)}\right)<\infty.$$

This is equivalent to (10), and completes the proof. □

6 Proof of Theorem 2

We represent the r-times integrated Wiener process W_r through a white-noise integral representation

$$W_r(t) := \int_0^1 \frac{(t-u)_+^r}{r!}\, \mathrm{d}\, W(u), \tag{27}$$

where the integration is carried over a standard Wiener process W defined over $[0,1]$. Clearly,

$$\begin{aligned} \mathbb{E}\|W_r\|_2^2 &= \sum_{j=1}^{\infty} \lambda_{j,r}^{\mathrm{W}} = \int_0^1 K_{1,r}^{\mathrm{W}}(t,t)\,\mathrm{d}t \\ &= \int_0^1 \left(\int_0^t \frac{(t-u)^{2r}}{r!^2}\,\mathrm{d}u \right) \mathrm{d}t = \int_0^1 \frac{t^{2r+1}}{(2r+1)r!^2}\,\mathrm{d}t = \frac{1}{(2r+2)(2r+1)r!^2}. \end{aligned} \tag{28}$$

We now supply a lower bound on the sum $\sum_{j=2}^{\infty} \lambda_{j,r}^{\mathrm{W}}$. To do this, we approximate W_r by

$$V_{r,1}(t) := t^r\, W_r(1) = \frac{1}{r!} \int_0^1 t^r (1-u)^r\, \mathrm{d}W(u) \qquad \text{for all} \quad t \in [0,1].$$

The process $V_{r,1}$ is of rank 1 since $V_{r,1}(t) := \xi_1(\omega)\psi_1(t)$, where $\psi_1(t) = t^r/r!$ and $\xi_1(\omega) = \int_0^1 (1-u)^r dW(u)$. We now prove the following lemma.

Lemma 4 *For any $r > 1$ we have*

$$\mathbb{E}|W_r(t) - V_{r,1}(t)|^2 \le \frac{1}{r!}\,\frac{3r^2}{(2r-2)^3}\, t^{2r-2}(1-t)^2 \qquad \textit{for all} \quad t \in [0,1], \tag{29}$$

and

$$\mathbb{E}||W_r - V_{r,1}||_2^2 \le \frac{1}{r!^2}\,\frac{6r^2}{(2r-2)^6}. \tag{30}$$

Before we prove the lemma, we stress that the order of the right hand side in (30) is smaller than that of $\mathbb{E}||W_r||_2^2$. This means that $V_{r,1}$ incorporates the essential part of W_r for large r.

Proof of Lemma 4. Let $\mathbf{1}_{\{0\le u\le t\}}$ be the characteristic function of $[0,t]$, i.e., $\mathbf{1}_{\{0\le u\le t\}} = 1$

for $u \in [0,t]$ and $\mathbf{1}_{\{0\le u\le t\}} = 0$ for $u \notin [0,t]$. We have

$$\begin{aligned}
\mathbb{E}|W_r(t) - V_{r,1}(t)|^2 &= \frac{1}{r!^2}\int_0^1 \left[t^r(1-u)^r - (t-u)^r \mathbf{1}_{\{0\le u\le t\}}\right]^2 \mathrm{d}u \\
&= \frac{t^{2r}}{r!^2}\int_0^t (1-u)^{2r}\left[1 - \left(\frac{t-u}{t(1-u)}\right)^r\right]^2 \mathrm{d}u + \frac{t^{2r}}{r!^2}\int_t^1 (1-u)^{2r}\,\mathrm{d}u \\
&= \frac{t^{2r}}{r!^2}\int_0^t (1-u)^{2r}\left[1 - \left(1 - \frac{(1-t)u}{t(1-u)}\right)^r\right]^2 \mathrm{d}u + \frac{t^{2r}}{r!^2}\int_t^1 (1-u)^{2r}\,\mathrm{d}u \\
&:= \frac{t^{2r}}{r!^2}[I_1 + I_2].
\end{aligned}$$

For I_1, we use an elementary bound $0 \le 1-(1-h)^r \le rh$ and get

$$\begin{aligned}
I_1 &\le \int_0^t (1-u)^{2r} r^2 \frac{(1-t)^2u^2}{t^2(1-u)^2}\,\mathrm{d}u \\
&= r^2(1-t)^2 t^{-2}\int_0^t (1-u)^{2r-2}u^2\,\mathrm{d}u \\
&\le r^2(1-t)^2 t^{-2}\int_0^\infty \exp(-(2r-2)u)u^2\,\mathrm{d}u \\
&= \frac{2r^2}{(2r-2)^3}\,(1-t)^2t^{-2}.
\end{aligned}$$

On the other hand,

$$I_2 = \int_0^{1-t} v^{2r}\,\mathrm{d}v = \frac{(1-t)^{2r+1}}{2r+1} \le \frac{r^2}{(2r-2)^3}\,(1-t)^2t^{-2}.$$

By summing up we obtain

$$\mathbb{E}|W_r(t) - V_{r,1}(t)|^2 \le \frac{1}{r!^2}\,\frac{3r^2}{(2r-2)^3} t^{2r-2}(1-t)^2,$$

as claimed in the first estimate of the lemma. The second claim is obtained by a simple

integration:

$$
\begin{aligned}
\mathbb{E}||W_r - V_{r,1}||_2^2 &= \int_0^1 \mathbb{E}|W_r(t) - V_{r,1}(t)|^2 \,\mathrm{d}t \\
&\le \frac{1}{r!^2}\,\frac{3r^2}{(2r-2)^3}\int_0^1 t^{2r-2}(1-t)^2 \,\mathrm{d}t \\
&= \frac{1}{r!^2}\,\frac{3r^2}{(2r-2)^3}\int_0^1 (1-t)^{2r-2}t^2 \,\mathrm{d}t \\
&\le \frac{1}{r!^2}\,\frac{3r^2}{(2r-2)^3}\int_0^\infty \exp(-(2r-2)t)t^2 \,\mathrm{d}t \\
&= \frac{1}{r!^2}\,\frac{6r^2}{(2r-2)^6}.
\end{aligned}
$$

as claimed. $\square$

From Lemma 4 we conclude that

$$
\sum_{j=2}^{\infty} \lambda_{j,r}^{\mathrm{W}} = \inf_{V \text{ is rank one}} \mathbb{E}||W_r - V||_2^2 \le \mathbb{E}||W_r - V_r||_2^2 \le \frac{C}{r!^2\, r^4}.
$$

This fact and (28) yield

$$
\lambda_{1,r}^{\mathrm{W}} = \frac{1}{r!^2}\left(\frac{1}{(2r+2)(2r+1)} + O(r^{-4})\right),
$$

as claimed in Theorem 2.

We now proceed to estimates on the second largest eigenvalue $\lambda_{2,r}^{\mathrm{W}}$ for large r. Obviously,

$$
\lambda_{2,r}^{\mathrm{W}} \le \sum_{j=2}^{\infty} \lambda_{j,r}^{\mathrm{W}} = \mathcal{O}\left(\frac{1}{r!^2\, r^4}\right). \tag{31}
$$

We now show that the last bound is essentially sharp. To do this we approximate W_r by

$$
V_{r,2}(t) := \frac{1}{r!}\int_0^1 \left[t^r(1-u)^r - rt^{r-1}(1-t)u(1-u)^{r-1}\right] \mathrm{d}W(u) \qquad \text{for all} \quad t \in [0,1].
$$

The process $V_{r,2}$ is of rank 2 since

$$
V_{r,2}(t) = \xi_1(\omega)\psi_1(t) - r\xi_2(\omega)\psi_2(t),
$$

where

$$\xi_1(\omega) = \int_0^1 (1-u)^r \,\mathrm{d}W(u) \qquad \text{and} \qquad \psi_1(t) = \frac{t^r}{r!},$$

$$\xi_2(\omega) = \int_0^1 u\,(1-u)^{r-1} \,\mathrm{d}W(u) \qquad \text{and} \qquad \psi_2(t) = \frac{t^{r-1}(1-t)}{r!}.$$

Note that the term $\xi_1\psi_1$ coming from rank 1 approximation is dominating in the rank 2 approximation, since

$$\mathbb{E}\,\xi_1^2||\psi_1||_2^2 = \int_0^1 (1-u)^{2r}\,\mathrm{d}u \cdot \frac{1}{r!^2} \cdot \int_0^1 t^{2r} dt = \frac{1}{r!^2}\frac{1}{(2r+1)^2} \approx \frac{1}{r!^2}\,r^{-2},$$

while for the correction term $r\xi_2\psi_2$ we have

$$r^2\mathbb{E}\,\xi_2^2||\psi_2||_2^2 = r^2\int_0^1 u^2(1-u)^{2r-2}\,\mathrm{d}u \cdot \frac{1}{r!^2} \cdot \int_0^1 t^{2r-2}(1-t)^2 dt \approx \frac{1}{r!^2} r^{-4}.$$

A careful analysis shows that the second eigenvalue of the covariance operator of $V_{2,r}$ is also of order $\frac{1}{r!^2}r^{-4}$. In other words, there exists a positive C independent of r such that

$$\inf_{V \text{ is rank one}} \mathbb{E}||V_{r,2} - V||_2^2 \geq \frac{C}{r!^2\,r^4}. \tag{32}$$

We now estimate how well $V_{r,2}$ approximates W_r.

Lemma 5 *For any $r > 2$ we have*

$$\mathbb{E}|W_r(t) - V_{r,2}(t)|^2 \leq \frac{1}{r!^2}\,\frac{14r^2(r-1)^2}{(2r-4)^5}\,t^{2r-4}(1-t)^4 \qquad \textit{for all} \quad t \in [0,1], \tag{33}$$

and

$$\mathbb{E}||W_r - V_{r,2}||_2^2 \leq \frac{1}{r!^2}\,\frac{24\cdot 14\cdot r^2(r-1)^2}{(2r-4)^{10}} = \mathcal{O}\left(\frac{1}{r!^2\,r^6}\right). \tag{34}$$

The proofs of (33) and (34) repeat (mostly, but not entirely) line by line those of Lemma 4 but we provide them for the sake of completeness. These proofs also clearly indicate how higher order approximations can be handled. As in Lemma 4 we again stress that the the order of the right hand side in (34) is smaller than the rank 1 approximation error computed in (30). Therefore, rank 2 approximation $V_{r,2}$ performs much better than rank 1 approximation $V_{r,1}$ for approximation of W_r when r is large.

Proof of Lemma 5. Let $a := \mathbb{E}|W_r(t) - V_{r,2}(t)|^2$. We have

$$
\begin{aligned}
a &= \frac{1}{r!^2}\int_0^1 \left[t^r(1-u)^r - rt^{r-1}(1-t)u(1-u)^{r-1} - (t-u)^r \mathbf{1}_{\{0\le u\le t\}}\right]^2 \mathrm{d}u \\
&= \frac{t^{2r}}{r!^2}\int_0^t (1-u)^{2r}\left[1 - \frac{r(1-t)u}{t(1-u)} - \left(\frac{t-u}{t(1-u)}\right)^r\right]^2 \mathrm{d}u \\
&\quad + \frac{t^{2r}}{r!^2}\int_t^1 \left((1-u)^r - \frac{r(1-t)u}{t}(1-u)^{r-1}\right)^2 \mathrm{d}u \\
&= \frac{t^{2r}}{r!^2}\int_0^t (1-u)^{2r}\left[1 - \frac{r(1-t)u}{t(1-u)} - \left(1 - \frac{(1-t)u}{t(1-u)}\right)^r\right]^2 \mathrm{d}u \\
&\quad + \frac{t^{2r}}{r!^2}\int_t^1 \left((1-u)^r - \frac{r(1-t)u}{t}(1-u)^{r-1}\right)^2 \mathrm{d}u =: \frac{t^{2r}}{r!^2}\,[I_1 + I_2].
\end{aligned}
$$

For I_1, we use an elementary bound $0 \ge 1 - rh - (1-h)^r \ge -\frac{r(r-1)}{2}h^2$ and get

$$
\begin{aligned}
I_1 &\le \int_0^t (1-u)^{2r}\left(\frac{r(r-1)}{2}\cdot\frac{(1-t)^2u^2}{t^2(1-u)^2}\right)^2 \mathrm{d}u \\
&= \frac{r^2(r-1)^2}{4}(1-t)^4 t^{-4}\int_0^t (1-u)^{2r-4}u^4\,\mathrm{d}u \\
&\le \frac{r^2(r-1)^2}{4}(1-t)^4 t^{-4}\int_0^\infty \exp(-(2r-4)u)u^4\,\mathrm{d}u \\
&= \frac{6r^2(r-1)^2}{(2r-4)^5}(1-t)^4 t^{-4}.
\end{aligned}
$$

On the other hand, we can give the following, rather crude, estimate for I_2. Note that for $u > t$ and $r > 1$ we have

$$
\frac{r(1-t)u}{t}(1-u)^{r-1} = r\cdot\frac{(1-t)u}{t(1-u)}\cdot(1-u)^r \ge (1-u)^r.
$$

Therefore,

$$
\begin{aligned}
I_2 &\leq \int_t^1 \left(\frac{r(1-t)u}{t}(1-u)^{r-1} \right)^2 \mathrm{d}u \\
&\leq \frac{r^2(1-t)^4}{t^4} \int_t^1 u^2(1-u)^{2r-4}\,\mathrm{d}u \\
&\leq \frac{r^2(1-t)^4}{t^4} \int_0^\infty u^2 \exp(-(2r-4)u)\,\mathrm{d}u \\
&= \frac{2r^2(1-t)^4}{(2r-4)^3 t^4} \leq \frac{8r^2(r-1)^2}{(2r-4)^5}(1-t)^4 t^{-4}.
\end{aligned}
$$

By summing up, we obtain

$$
\mathbb{E}|W_r(t) - V_{r,2}(t)|^2 \leq \frac{1}{r!^2} \frac{14r^2(r-1)^2}{(2r-4)^5}(1-t)^4 t^{2r-4},
$$

as claimed in the first estimate of the lemma. The second claim is obtained by a simple integration:

$$
\begin{aligned}
\mathbb{E}\,||W_r - V_{r,2}||_2^2 &= \int_0^1 \mathbb{E}|W_r(t) - V_{r,2}(t)|^2\,\mathrm{d}t \\
&\leq \frac{1}{r!^2} \frac{14r^2(r-1)^2}{(2r-4)^5} \int_0^1 (1-t)^4 t^{2r-4}\,\mathrm{d}t \\
&= \frac{1}{r!^2} \frac{14r^2(r-1)^2}{(2r-4)^5} \int_0^1 (1-t)^{2r-4} t^4\,\mathrm{d}t \\
&\leq \frac{1}{r!^2} \frac{14r^2(r-1)^2}{(2r-4)^5} \int_0^\infty \exp(-(2r-4)t) t^4\,\mathrm{d}t \\
&= \frac{1}{r!^2} \frac{24 \cdot 14r^2(r-1)^2}{(2r-4)^{10}},
\end{aligned}
$$

as claimed. □

From Lemma 5 we easily estimate $\lambda_{2,r}^{\mathrm{W}}$. Let $\zeta\eta_1 := \zeta(\omega)\eta_1(t)$ be the first term of Karhunen-Loève expansion for W_r. Then

$$
\begin{aligned}
\frac{C}{r!^2\, r^4} &\overset{\text{by (32)}}{\leq} \mathbb{E}\,||V_{r,2} - \zeta\eta_1||_2^2 \\
&= \mathbb{E}\,||(V_{r,2} - W_r) + (W_r - \zeta\eta_1)||_2^2 \\
&\leq 2\,\mathbb{E}\,||V_{r,2} - W_r||_2^2 + 2\,\mathbb{E}\,||W_r - \zeta\eta_1)||_2^2 \\
&= 2\,\mathbb{E}\,||V_{r,2} - W_r||_2^2 + 2\lambda_{2,r}^{\mathrm{W}} + 2\sum_{j=3}^\infty \lambda_{j,r}^{\mathrm{W}}.
\end{aligned}
$$

Since $V_{r,2}$ is a process of rank 2, we also have

$$\sum_{j=3}^{\infty} \lambda_{j,r}^{\mathrm{W}} = \inf_{V \text{ of rank two}} \mathbb{E}\,||W_r - V||_2^2 \leq \mathbb{E}\,||W_r - V_{r,2}||_2^2. \tag{35}$$

For future use, we combine this with (34) and get

$$\sum_{i=3}^{\infty} \lambda_{i,r}^{\mathrm{W}} \leq \frac{C_1}{r!^2\, r^6}. \tag{36}$$

Furthermore, (35) immediately yields

$$\frac{C}{r!^2\, r^4} \leq 4\mathbb{E}||V_{r,2} - W_r||_2^2 + 2\lambda_{2,r} \overset{\text{by (34)}}{\leq} \frac{C_1}{r!^2\, r^6} + 2\lambda_{2,r}^{\mathrm{W}}.$$

This provides a lower bound for $\lambda_{2,r}^{\mathrm{W}}$ and together with (31) proves that

$$\lambda_{2,r}^{\mathrm{W}} = \Theta\left(\frac{1}{r!^2\, r^4}\right), \tag{37}$$

as claimed.

We are ready to prove the last assertion of Theorem 2. To simplify notation, let $\lambda_{j,r} = \lambda_{j,r}^{\mathrm{W}}$. We split the series $\sum_{j=3}^{\infty} \lambda_{j,r}$ into two pieces - a long but finite initial part and a tail. Let $M > 2$ and $\tau \in [\tau_0, 1]$ with $\tau_0 \in (\frac{3}{5}, 1]$. Consider the initial part including $j = 3, 4, \ldots, \lceil r^M \rceil$. Using Hölder's inequality we obtain

$$\begin{aligned}
\sum_{j=3}^{\lceil r^M \rceil} \lambda_{j,r}^{\tau} \quad &\leq \quad \left(\sum_{j=3}^{\lceil r^M \rceil} \lambda_{j,r}\right)^{\tau} \left(\sum_{j=3}^{\lceil r^M \rceil} 1\right)^{1-\tau} \\
&\overset{\text{by (36)}}{\leq} \quad \left(\frac{C_1}{r!^2\, r^6}\right)^{\tau} r^{M(1-\tau)} = r^{-2\tau} \left(\frac{C_1}{r!^2\, r^4}\right)^{\tau} r^{M(1-\tau)} \\
&\overset{\text{by (37)}}{\leq} \quad C\lambda_{2,r}^{\tau} r^{-2\tau + M(1-\tau)} \leq C\lambda_{2,r}^{\tau} r^{-2\tau_0 + M(1-\tau_0)}.
\end{aligned}$$

Since C can be taken independent of τ, for some $h > 0$ we have

$$\sup_{\tau \in [\tau_0, 1]} \frac{\sum_{j=3}^{\lceil r^M \rceil} \lambda_{j,r}^{\tau}}{\lambda_{2,r}^{\tau}} = \mathcal{O}(r^{-h}), \quad \text{as } r \to \infty,$$

as long as

$$M < \frac{2\tau_0}{1-\tau_0}. \tag{38}$$

For the tail estimation of the eigenvalue series $\sum_{j=\lceil r^M \rceil+1}^{\infty} \lambda_{j,r}$ we use *approximation numbers* (or *linear widths*, in other terminology).

We need to recall the definition and few basic properties which we will use in the sequel. Let $A : B_1 \to B_2$ be a bounded linear operator acting between two Banach spaces. The approximation number $a_n(A)$ for $n \geq 1$ is defined as

$$a_n(A) := \inf \left\{ \|A - A_n\| \;\middle|\; A_n : B_1 \to B_2 \quad \text{with} \quad \operatorname{rank}(A_n) < n \right\}. \tag{39}$$

The following properties of $a_n(A)$ are well known, see [14].

- the sequence $\{a_n(A)\}_{n \in \mathbb{N}}$ is non-increasing,
- for the adjoint operator A^* we have
$$a_n(A) = a_n(A^*), \tag{40}$$
- multiplicative property: for $A_1 : B_1 \to B_2$ and $A_2 : B_2 \to B_3$ we have
$$a_{n+m-1}(A_2 A_1) \leq a_n(A_2)\, a_m(A_1) \qquad \text{for all} \quad n, m \in \mathbb{N}, \tag{41}$$
- if $A : H \to H$ is a self-adjoint compact operator acting for a Hilbert space H with the non-increasing eigenvalues $\{\lambda_n\}$ then
$$a_n(A) = \lambda_n. \tag{42}$$

We will study approximation numbers for integration operators. Let $I : L_2[0,1] \to L_2[0,1]$ be the conventional integration operator

$$(Ix)(t) := \int_0^t x(s)\, \mathrm{d}s \qquad \text{for all} \quad t \in [0,1].$$

Let I^r denote the r-th iteration of I for $r \geq 1$. It is easy to check by induction that

$$\begin{aligned}
(I^r x)(t) &= \int_0^t \frac{(t-s)^{r-1}}{(r-1)!}\, x(s)\, \mathrm{d}s \qquad \text{for all} \quad t \in [0,1], \\
([I^r]^* x)(t) &= \int_t^1 \frac{(s-t)^{r-1}}{(r-1)!}\, x(s)\, \mathrm{d}s \qquad \text{for all} \quad t \in [0,1], \\
(I^r\, [I^r]^*)(t) &= \int_0^1 \left(\int_0^{\min(s,t)} \frac{(s-u)_+^{r-1}}{(r-1)!} \, \frac{(t-u)_+^{r-1}}{(r-1)!}\, \mathrm{d}u \right) x(s)\, \mathrm{d}s \qquad \text{for all} \quad t \in [0,1].
\end{aligned}$$

This shows that

$$C^{\mathrm{w}}_{1,r} = I^{r+1}\,(I^{r+1})^*.$$

We are interested in the approximation numbers of I^r. For $r=0$, it is well known that for some positive C we have

$$a_n(I) \le C\,n^{-1} \quad \text{for all} \quad n \in \mathbb{N}, \tag{43}$$

see [2], pp. 118–119. We will extend this estimate for I^r with an arbitrary r. Although the constant we get is certainly not optimal, it suffices for our needs.

Lemma 6 *We have*

$$a_n(I^r) \le C^{\,r}\,(2r)^{2r}\,n^{-r} \qquad \textit{for all} \quad n, r \in \mathbb{N}, \tag{44}$$

where C is a constant from (43).

Proof of Lemma 6. Let

$$B_p := 2^{\,p\,2^p} \qquad \text{for all} \quad p = 0, 1, 2, \dots .$$

We will first prove by induction on p that for any integer $p \ge 0$ we have

$$a_n(I^r) \le C^{\,r}\,B_p\,n^{-r} \qquad \text{for all} \quad n \ge 1 \ \text{ and } \ r \in [2^{p-1}, 2^p]. \tag{45}$$

For $p=0$ this fact is equivalent to (43). Assume that (45) holds for some integer p. Take any integer $r \in [2^p, 2^{p+1}]$ and write it as $r = r' + r''$ with $2^{p-1} \le r_1, r_2 \le 2^p$. By using $I^r = I^{r_1} I^{r_2}$ and the multiplicative property (41), we get for an odd index $2n-1$

$$\begin{aligned}
a_{2n-1}(I^r) &= a_{2n-1}(I^{r_1} I^{r_2}) \le a_n(I^{r_1})\,a_n(I^{r_2}) \\
&\le C^{r_1} B_p n^{-r_1} \cdot C^{r_2} B_p n^{-r_2} = C^r\,B_p^2\,n^{-r} \\
&= C^r\,B_p^2\,2^r\,(2n)^{-r} \le C^r\,[B_p^2 2^{2^{p+1}}]\,(2n)^{-r} \\
&= C^r 2^{\,2p2^p + 2^{p+1}}\,(2n)^{-r} = C^r\,2^{(p+1)2^{p+1}}\,(2n)^{-r} \\
&= C^r\,B_{p+1}\,(2n)^{-r} \le C^r B_{p+1}\,(2n-1)^{-r}.
\end{aligned}$$

For an even index $2n$ we simply have

$$a_{2n}(I^r) \le a_{2n-1}(I^r) \le C^r B_{p+1}\,(2n)^{-r}.$$

Therefore, (45) is proved by induction.

For r and p as in (45), we have $B_p = (2^p)^{2^p} \le (2r)^{2r}$. Hence, (44) follows from (45). $\square$

We now relate approximation numbers $a_n(I^r)$ to the eigenvalues $\lambda_{j,r}$ of the operator $C_{1,r}^{\mathrm{W}} = I^{r+1}(I^{r+1})^*$. We have

$$\begin{aligned}\lambda_{2j,r} &\le \lambda_{2j-1,r} \overset{\text{by (42)}}{=} a_{2j-1}(I^{r+1}(I^{r+1})^*)\\ &\overset{\text{by (41)}}{\le} a_j(I^{r+1})a_j((I^{r+1})^*) \overset{\text{by (40)}}{=} a_j(I^{r+1})^2\\ &\overset{\text{by (44)}}{\le} C^{2(r+1)}(2(r+1))^{4(r+1)}j^{-2(r+1)}.\end{aligned}$$

This can be written as

$$\lambda_{j,r} \le C_1^r\, r^{4(r+1)}\, j^{-2(r+1)} \qquad \text{for all} \quad r,j\in\mathbb{N}.$$

Take a (small) positive α. Consider r so large that $r \ge C_1^{1/\alpha}$ and $2(r+1)\tau > 1$. Then again for $\tau\in[\tau_0,1]$ we can sum up

$$\begin{aligned}\sum_{j=\lceil r^M\rceil+1}^{\infty} \lambda_{j,r}^{\tau} &\le C_1^{r\tau}\, r^{4(r+1)\tau} \sum_{j=\lceil r^M\rceil+1}^{\infty} j^{-2(r+1)\tau}\\ &\le r^{(4+\alpha)r\tau+4\tau}\int_{r^M}^{\infty} x^{-2(r+1)\tau}\,\mathrm{d}x\\ &= \frac{r^{r\tau(4+\alpha-2M)}\ r^{4\tau+M(1-2\tau)}}{2(r+1)\tau-1}.\end{aligned}$$

We relate the last estimate to $\lambda_{2,r} = \Theta(1/(r!^2 r^4))$. Since $r! = r^{r+1/2}\, e^{-r}\sqrt{2\pi}(1+o(1))$ by Stirling's formula, we have

$$\lambda_{2,r} = \frac{e^{2r}}{2\pi\, r^{2r+5}}\,(1+o(1)) \qquad \text{as} \quad r\to\infty.$$

Therefore

$$\sum_{j=\lceil r^M\rceil+1}^{\infty} \lambda_{j,r}^{\tau} = \mathcal{O}\left(\lambda_{2,r}^{\tau}\,\frac{r^{-r\tau(2M-6-\alpha)}\ r^{9\tau+M(1-2\tau)}}{2(r+1)\tau-1}\,e^{-2r\tau}\right) = \mathcal{O}\left(\lambda_{2,r}^{\tau}\, r^{-r\tau(2M-6-\alpha)}\right),$$

where the factors in the big $\mathcal{O}$ notation are independent of r,τ and α.

Assume that $M>3$. Then we can take a positive α such that $2M-6-\alpha>0$ and get

$$\sup_{\tau\in[\tau_0,1]} \frac{\sum_{j=\lceil r^M\rceil+1}^{\infty}\lambda_{j,r}^{\tau}}{\lambda_{2,r}^{\tau}} = \mathcal{O}(r^{-h}), \quad \text{as } r\to\infty.$$

Hence,

$$\sup_{\tau\in[\tau_0,1]} \frac{\sum_{j=1}^\infty \lambda_{j,r}^\tau}{\lambda_{2,r}^\tau} = \mathcal{O}(r^{-h}) \quad \text{as } r\to\infty,$$

assuming (38) holds for some $M>3$. It is easy to see that such a number M exists since $\tau > \frac{3}{5}$. This completes the proof. □

7 Proof of Theorem 3

As in the Euler case, we begin with polynomial tractability. We now need to show that

$$\text{PT} \;\Rightarrow\; \liminf_k \frac{r_k}{k^s} > 0 \;\Rightarrow\; \text{SPT} \;\Rightarrow\; \text{PT}.$$

Observe that for $\lambda_{d,j} = \lambda_{d,j}^{\mathrm{W}}$ and $\tau\in(0,1)$, the expression in (15) is now

$$a_d := \frac{\left(\sum_{j=1}^\infty \lambda_{d,j}^\tau\right)^{1/\tau}}{\sum_{j=1}^\infty \lambda_{d,j}} = \prod_{k=1}^d \frac{\left(1+(\lambda_{2,r_k}/\lambda_{1,r_k})^\tau + \sum_{j=3}^\infty (\lambda_{j,r_k}/\lambda_{1,r_k})^\tau\right)^{1/\tau}}{1+\lambda_{2,r_k}/\lambda_{1,r_k} + \sum_{j=3}^\infty \lambda_{j,r_k}/\lambda_{1,r_k}}. \tag{46}$$

Since $\lambda_{j,r_k} = \Theta(j^{-2(r_k+1)})$ as $j\to\infty$, with the factors in the Θ notation depending on r_k, then a_d is finite iff $2(r_k+1)\tau>1$ for all r_k. Then $r_k \ge r_1$ implies that we need to consider $\tau\in(\frac{1}{2r_1+2},1)$.

Assume that we have polynomial tractability. Then $a_d \le C\, d^q$. Each ratio in the product (46) is strictly larger than one. This implies that $\lim_{k\to\infty} r_k = \infty$.

Note that we can estimate a_d from below by dropping the sums over j. Then

$$\prod_{k=1}^d \frac{\left(1+(\lambda_{2,r_k}/\lambda_{1,r_k})^\tau\right)^{1/\tau}}{1+2\lambda_{2,r_k}/\lambda_{1,r_k}} < C\, d^q\ .$$

Taking logarithms and using the asymptotic formulas for λ_{1,r_k} and λ_{2,r_k} from Theorem 2 yield

$$\sup_d \frac{1}{\ln_+ d}\ \sum_{k=1}^d r_k^{-2\tau} < \infty.$$

Since $d\, r_d^{-2\tau} \le \sum_{k=1}^d r_k^{-2\tau}$ we get $r_d^{-2\tau} = \mathcal{O}(d^{-1}\ \ln_+ d)$ and there exists $\delta>0$ such that

$$r_d \ge \delta \left(\frac{d}{\ln_+ d}\right)^{1/(2\tau)} \qquad \text{for all} \quad d\in\mathbb{N}.$$

Letting $s \in (\frac{1}{2}, \frac{1}{2\tau})$ we obtain

$$\liminf_{k\to\infty} \frac{r_k}{k^s} > 0, \tag{47}$$

as claimed.

Assume now that (47) holds for some $s > \frac{1}{2}$. For $\tau \in (\max(\frac{3}{5}, \frac{1}{2s}), 1]$ we can use the last assertion of Theorem 2 to conclude that

$$\sup_d a_d = \prod_{k=1}^{\infty} \frac{\left(1+\mathcal{O}(r_k^{-2\tau})\right)^{1/\tau}}{1+\mathcal{O}(r_k^{-2})} \le \exp\left\{\mathcal{O}\left(\sum_{k=1}^{\infty} r_k^{-2\tau}\right)\right\} = \exp\left\{\mathcal{O}\left(\sum_{k=1}^{\infty} k^{-2s\tau}\right)\right\} < \infty. \tag{48}$$

By criterion (15) this implies strong polynomial and obviously polynomial tractability.

We turn to weak tractability. Assume that $\lim_{k\to\infty} r_k = \infty$. We verify the analogue of (19) for $\tau \in (\frac{3}{5}, 1)$. From Theorem 2 we have

$$b_d := \frac{1}{d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda^{\mathrm{W}}_{j,r_k}}{\lambda^{\mathrm{W}}_{1,r_k}}\right)^{\tau} = \frac{1}{d} \sum_{k=1}^{d} \mathcal{O}\left(r_k^{-2\tau}\right) = \mathcal{O}\left(\frac{1}{d} \sum_{k=1}^{d} r_k^{-2\tau}\right).$$

Clearly, $\lim_k r_k^{-2\tau} = 0$ implies $\lim_d b_d = 0$, which yields weak tractability.

Let $r = \lim_{k\to\infty} r_k < \infty$. Then proceeding exactly as for the Euler case, we can show that $n^{\mathrm{W}}(\varepsilon, d)$ is an exponential function of d which contradicts weak tractability and completes this part of the proof.

We finally consider quasi-polynomial tractability. The proof is similar to the proof for the Euler case and we only sketch it. We need to study (20) and (25) for the Wiener eigenvalues. For (20) we take $\delta = \frac{1}{2}$ and $\tau_0 \in (\frac{3}{5}, 1)$. Let us chose d_0 such that $1 - \frac{1}{2\ln d_0} \in [\tau_0, 1]$. Then for all such $d \ge d_0$ we have $\tau_d := 1 - 1/(2\ln d) \in [\tau_0, 1]$ and we can use the result on the uniform convergence presented in the last assertion of Theorem 2 with respect now to d. Let denote $Q_k := \frac{\lambda_{2,r_k}}{\lambda_{1,r_k}}$. We obtain

$$\begin{aligned}
\frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\delta/\ln_+ d}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\delta/\ln_+ d}} &= \prod_{k=1}^{d} \frac{1 + Q_k^{1-\frac{1}{2\ln d}} + \sum_{j=3}^{\infty} \left(\frac{\lambda_{j,r_k}}{\lambda_{1,r_k}}\right)^{1-\frac{1}{2\ln d}}}{\left(1 + Q_k + \sum_{j=3}^{\infty} \frac{\lambda_{j,r_k}}{\lambda_{1,r_k}}\right)^{1-\frac{1}{2\ln d}}} \\
&\le \mathcal{O}(1) \prod_{k=d_0}^{d} \frac{1 + Q_k^{1-\frac{1}{2\ln d}} \left(1 + o(r_k^{-h})\right)}{(1+Q_k)^{1-\frac{1}{2\ln d}}},
\end{aligned}$$

with absolute constants as pre-factors in the $\mathcal{O}(\cdot)$ notation.

Suppose that (13) holds. Then $\lim_k r_k = \infty$ and

$$\prod_{k=d_0}^{d} (1+Q_k)^{\frac{1}{2\ln d}} \le \exp\left(\frac{2}{\ln d}\sum_{k=d_0}^{d} Q_k\right) \le \exp\left(\frac{C}{\ln d}\sum_{k=d_0}^{d} r_k^{-2}\right)$$

is uniformly bounded in d. The factor $\prod_{k=d_0}^{d} \frac{1+Q_k^{1-\frac{1}{2\ln d}}\left(1+o(r_k^{-h})\right)}{1+Q_k}$ can be analyzed exactly as for the Euler case. By using $Q_k = \Theta(r_k^{-2})$, we have

$$\frac{1+Q_k^{1-\frac{1}{2\ln d}}\left(1+o(r_k^{-h})\right)}{1+Q_k} \le 1 + d^{-3/2} + C\,(1+r_k)^{-2}\left(\frac{\ln_+ r_k}{\ln d} + o(r_k^{-h})\right).$$

Recall that assumption (13) yields $r_k^{-2} = \mathcal{O}(\frac{\ln k}{k})$, hence

$$\prod_{k=d_0}^{d} \frac{1+Q_k^{1-\frac{1}{2\ln d}}\left(1+o(r_k^{-h})\right)}{1+Q_k} \le \exp\left(\sum_{k=d_0}^{d}\left(d^{-3/2} + C\,(1+r_k)^{-2}\left[\frac{\ln_+ r_k}{\ln d} + r_k^{-h}\right]\right)\right)$$

is also uniformly bounded in d. This means that (13) implies quasi-polynomial tractability.

Suppose now that quasi-polynomial tractability holds. Then we use (25) and its consequence (26), which is equivalent to (13). This completes the proof. □

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